MATH 147 Review: Line Integrals

Facts to Know

If \( f \) is a real-valued function defined on a smooth curve \( C \subseteq \mathbb{R}^2 \), then

\[
\int_C f(x, y) dx = \int_a^b f(x(t), y(t)) x'(t) \, dt
\]

\[
\int_C f(x, y) dy = \int_a^b f(x(t), y(t)) y'(t) \, dt
\]

Here \( x(t), y(t), t \in [a, b] \) is a parametrization of \( C \) (with a given orientation).

Examples

1. Evaluate \( \int_C (2 + x^2 y) \, dy \), where \( C \) is the upper half of the unit circle \( x^2 + y^2 = 1 \) with a clockwise orientation.

\[
\begin{align*}
\gamma(t) &= 1 \cos(-t + \pi) + 0 & \quad t \in [0, \pi] \\
\gamma(t) &= 1 \sin(-t + \pi) + 0 \\
I &= \int_0^\pi (2 + \cos^2(-t+\pi) \sin(-t+\pi)) \cos(-t+\pi) \, dt \\
&= \int_0^\pi 2(-1) \cos(-t+\pi) + \cos^3(-t+\pi) \sin(-t+\pi) \, dt \\
&= \int_0^\pi (-2 \cos(-t+\pi)) + \int_0^\pi (-1) \cos^3(-t+\pi) \sin(-t+\pi) \, dt \\
&= \int_0^\pi \cos(-t+\pi) \, du \quad \text{where} \quad du = \cos(-t+\pi) \, dt \\
&= \int_{-1}^{+1} u^3 \, du = 0
\end{align*}
\]
2. Evaluate $\int_C 2x \, dx$, where $C$ consists of the arc $C_1$ of the line segment from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment $C_2$ from $(1, 1)$ to $(1, 2)$.

\[
\int_C 2x \, dx = \int_{C_1} 2x \, dx + \int_{C_2} 2x \, dx
= \int_0^1 2t \cdot 1 \, dt + \int_0^1 2 \cdot (1) \cdot 0 \, dt
= \int_0^1 2t \, dt
= t^2 \bigg|_0^1 = 1^2 - 0^2 = 1
\]